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BIFURCATION OF SELF-SIMILAR SOLUTIONS DESCRIBING A THERMOCAPILLARY FLOW OF A FLUID IN A THIN LAYER

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Thermocapillary flows of a fluid in a lamina with a rigid lower wall and a free upper surface, along which the temperature gradient is given in the radial direction, are investigated for large Marangoni numbers. Self-similar solutions which describe the axisymmetric flow regimes of a fluid without the circumferential velocity component are constructed numerically and asymptotically for a system of Prandtl equations. It is shown that a pair of new self-similar flow regimes of a fluid with rotation branches off from the regimes obtained. The new regimes ere calculated numerically and asymptotically.

1. The steady-state problem of a thermocapillary flow of an incompressible fluid in a thin horizontal layer bounded by a rigid wall S from below and by a free boundary Γ from above, on which a nonzero temperature gradient is given for small viscosity ($\nu \to 0$) and thermal-diffusivity ($\chi \to 0$) coefficients, is described by the system

$$(\boldsymbol{v}, \nabla)\boldsymbol{v} = -\rho^{-1}\nabla p + \nu\Delta\boldsymbol{v} + \boldsymbol{g}, \qquad \boldsymbol{v}\cdot\nabla T = \chi\Delta T, \quad \operatorname{div}\boldsymbol{v} = 0$$

with the boundary conditions

$$p = 2\nu\rho\boldsymbol{n}\Pi\boldsymbol{n} - \sigma(k_1 + k_2) + p_*, \qquad (r,\theta,z) \in \Gamma,$$

$$2\nu\rho[\Pi\boldsymbol{n} - (\boldsymbol{n}\Pi\boldsymbol{n})\boldsymbol{n}] = \nabla_\Gamma\sigma, \qquad \boldsymbol{v}\cdot\boldsymbol{n} = 0, \qquad (r,\theta,z) \in \Gamma,$$

$$T = T_\Gamma, \quad (r,\theta,z) \in \Gamma, \qquad \boldsymbol{v} = T - T_S = 0, \qquad (r,\theta,z) \in S.$$

Here $\boldsymbol{v} = (v_r, v_\theta, v_z)$ is the velocity vector, (r,θ,z) are the cylindrical coordinates, $\boldsymbol{g} = (0,0,-g_t)$, where g_t is the acceleration of gravity, T is the temperature, $\boldsymbol{n}$ is the unit vector of the external normal to a free boundary Γ, Π is the strain-rate tensor, k_1 and k_2 are the principal curvatures of the surface Γ, T_S is the wall temperature, p_* and T_Γ are the specified pressure and temperature on the free boundary, $\nabla_\Gamma = \nabla - (\boldsymbol{n},\nabla)\boldsymbol{n}$ is the gradient along Γ, and $\sigma = \sigma_0 - |\sigma_T|(T - T_*)$ is the surface-tension coefficient which linearly depends on the temperature, where σ_0, σ_T, and T_* are known constants. The axial-symmetry conditions mean that $\boldsymbol{v}$, p, and T do not depend on the circumferential coordinate θ.

Upon irregular heating of the free boundary Γ, surface tangent stresses arise on this boundary due to the thermocapillary Marangoni effect. These stresses lead to the formation of nonlinear boundary layers as $\nu \to 0$ and $\chi \to 0$.

We consider a thermocapillary flow of a fluid in a lamina whose thickness is of the order of the thickness of the boundary layer $O(\varepsilon)$. We note that the flows induced by the Marangoni effect in the layers of thickness $O(\varepsilon)$ which are limited by rigid and free boundaries were studied by the author in [1], and Anderson et al. [2] studied the flows in the layers between two rigid walls with the use of the Prandtl equations. Pukhnachev [3] constructed unsteady-state self-similar solutions which describe the flows of a fluid in the boundary layer near the free boundary.

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We consider a thermocapillary flow of a fluid in a thin viscous layer of thickness $O(\varepsilon)$ with the use a system of Prandtl equations [1]. We write this system in cylindrical coordinates, taking into account that $\boldsymbol{v}, p$, and T do not depend on the circumferential coordinate θ:

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = \nu \frac{\partial^2 v_r}{\partial z^2} - \frac{1}{\rho}\frac{\partial p'}{\partial r}, \qquad v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = \nu \frac{\partial^2 v_\theta}{\partial z^2},$$
$$\frac{\partial p'}{\partial z} = 0, \qquad \frac{\partial (r v_r)}{\partial r} + \frac{\partial (r v_z)}{\partial z} = 0, \qquad v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} = \chi \frac{\partial^2 T}{\partial z^2}. \tag{1.1}$$

Here $p' = p + \rho g_t z$. For system (1.1), the following boundary conditions at the rigid wall $z = 0$ and at the free boundary $z = \zeta$ are used in a boundary-layer approximation:

$$\rho\nu \frac{\partial v_r}{\partial z} = \frac{\partial \sigma}{\partial r}, \qquad \frac{\partial v_\theta}{\partial z} = 0, \qquad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad (z = \zeta),$$
$$p' = \rho g_t z - \sigma(k_1 + k_2) + 2\nu\rho \frac{\partial v_n}{\partial n} p_*, \qquad T = T_\Gamma \quad (z = \zeta), \tag{1.2}$$
$$v_r = v_\theta = v_z = 0, \qquad T = T_S \quad (z = 0).$$

The boundary conditions for $z = \zeta$ are the dynamic conditions for the tangent and normal stresses and the kinematic condition on the free boundary Γ. We note that the tangent stresses on Γ are caused by the Marangoni effect.

2. We construct a self-similar solution of system (1.1), (1.2) provided that the temperature on the free boundary depends only on the radial coordinate according to a quadratic law $T_\Gamma = 0.5 A_\Gamma r^2$. In this case, the tangent stresses on Γ act only in the radial direction and are absent in the circumferential direction. We present the self-similar solution of system (1.1)

$$v_r = r\Phi'(\xi)\nu h L^{-2}, \qquad v_\theta = r g(\xi)\nu h L^{-2}, \qquad v_z = 2\Phi(\xi)\nu h^2 L^{-1},$$
$$T = 0.5 r^2 A_\Gamma T_1(\xi), \qquad \frac{\partial p}{\partial r} = \rho q r \nu^2 L^{-2} h^2, \qquad \xi = 1 - \frac{z}{H}.$$

Here H is the thickness of the layer, $L = (\rho\nu^2 A_\Gamma^{-1}|\sigma_T|^{-1})^{1/3}$ is the scale of length, and $h = H/L$ is the dimensionless parameter. The boundary conditions on the free boundary are fulfilled if $\zeta = H = \text{const}$ and p_* depends on the coordinate r according to a quadratic law. We note that the free boundary is rectilinear in this approximation, and the layer has a constant thickness of the order $\nu^{2/3}$. Obviously, we have $\varepsilon = O(\nu^{2/3})$.

The self-similar solution obtained describes the thermocapillary flow of a fluid only near the axis of symmetry Oz and is not extended to the case of large values of the radial coordinate r.

The functions $\Phi(\xi)$, $g(\xi)$, and $T_1(\xi)$ and the parameter q are determined from the boundary-value problem

$$\Phi''' = \lambda(\Phi'^2 - 2\Phi\Phi'' - g^2 + q), \qquad g'' = 2\lambda(\Phi' g - \Phi g'), \qquad q' = 0,$$
$$T_1'' = 2\lambda \Pr(\Phi' T_1 - \Phi T_1'), \qquad \Phi(0) = 0, \quad \Phi''(0) = 1, \tag{2.1}$$
$$g'(0) = 0, \qquad T_1(0) = 1, \qquad \Phi(1) = \Phi'(1) = g(1) = T_1(1) = 0.$$

It is assumed that $\lambda = h^3$ is the specific parameter, Pr is the Prandtl number, and the tangent stresses on the free boundary are directed toward the axis of symmetry.

The solution of system (2.1), which describes fluid flow with a zero circumferential velocity component, is denoted by $\Phi_0(\xi, \lambda)$, $q_0(\lambda)$, and $g_0 \equiv 0$. For finite λ, this solution is obtained numerically. Curve 1 in Fig. 1 refers to the parameter $p_0 = -q_0 h^2$ versus the dimensionless thickness of the layer h. For small λ, expanding the function $\Phi_0(\xi, \lambda)$ into a power series of the variable ξ and retaining three terms, we find the following asymptotic values ($\lambda \to 0$):

$$\Phi_0 = -\xi(\xi - 1)^2/4 + o(1), \qquad q_0 = -1.5/\lambda + o(1). \tag{2.2}$$

For $h = 0.5$, the first three significant figures of the numerical and asymptotic values of q_0 coincide. For large λ, the solution is constructed by the method of stitching the asymptotic expansions:

$$\Phi_0 = 0.5\xi(\xi - 1) + \lambda^{-1/2} g_a(\eta) + O(\lambda^{-1}), \qquad q_0 = -1/4 + O(\lambda^{-1/2}) \quad (\lambda \to \infty).$$

Here $\eta = (1 - \xi)\lambda^{-1/2}$, and the function $g_a(\eta)$ is defined from the boundary-value problem

$$g_a''' = 2g_a g_a'' - g_a'^2 + \eta g_a'' + g_a', \qquad g_a(0) = g_a'(0) - 0.5 = g_a'(\infty) = 0.$$

A numerical calculation yields $g_a''(0) = 0.5478$.

3. We show that, for a certain λ, two symmetric solutions with a nonzero circumferential velocity component $v_\theta \neq 0$ bifurcates from the solution Φ_0, q_0. To do this, we first consider the eigenvalue problem obtained by linearizing problem (2.1) near the solution Φ_0, q_0:

$$\begin{gathered} f_1''' = \lambda(2\Phi_0' f_1 - 2\Phi_0 f_1'' - 2\Phi_0'' f_1 + q_1), \qquad g_1'' = 2\lambda(\Phi_0' g_1 - \Phi_0 g_1'), \qquad q_1' = 0, \\ f_1(0) = f_1''(0) = g_1'(0) = 0, \qquad f_1(1) = f_1'(1) = g_1(1) = 0. \end{gathered} \tag{3.1}$$

The resulting boundary-value problem (with allowance for the corresponding problem for the function T_1) was investigated numerically for finite values of λ and asymptotically for small and large λ. The calculations showed that, for finite values of λ, there is only one simple eigenvalue of $\lambda_0 = 11.222$ to which corresponds the eigenfunction $g_1 = \varphi(\xi)$, $f_1 = 0$, $q_1 = 0$, and $T_1 = 0$ with the normalization condition $\varphi(0) = 1$. On the segment $[0, 1]$, with increase in ξ, the positive function $\varphi(\xi)$ decreases monotonically from unity to zero. A study of problem (3.1) by the method of stitching the asymptotic expansions showed that the eigenvalues are absent as $\lambda \to \infty$. For small λ, a study by means of formulas (2.2) showed that problem (3.1) has no eigenvalues either. This problem has only one simple eigenvalue for all $\lambda \in (0, \infty)$.

We pass to the derivation of a bifurcation equation for the boundary-value problem (2.1) with the use of the method of [4] and by representing the solution in the form

$$\Phi(\xi, \lambda) = \Phi_0(\xi, \lambda) + \alpha f(\xi, \lambda, \alpha), \qquad g(\xi, \lambda) = \alpha G(\xi, \lambda, \alpha), \qquad q = q_0(\lambda) + \alpha Q(\lambda, \alpha), \tag{3.2}$$

where f, G, and Q are the desired functions and α is a parameter chosen in such a way that the condition $G = 1$ is satisfied for $\xi = 0$. We introduce the linear operators

$$\begin{gathered} L = D^3 - \lambda(2\Phi_0' D - 2\Phi_0 D^2 - 2\Phi_0'' I), \\ K = D^2 - 2\lambda(\Phi_0' I - \Phi_0 D), \quad L_0 = L, \quad K_0 = K \quad (\lambda = \lambda_0). \end{gathered}$$

Here $D = d/d\xi$ and I is the unit operator.

The functions f, G, and Q are determined from the nonlinear boundary-value problem

$$\begin{gathered} L_1(f, g, Q) \equiv Lf - \lambda Q - \lambda\alpha(f'^2 - 2ff'' - G^2) = 0, \\ K_1(f, G) \equiv KG - 2\lambda\alpha(f'G - fG') = 0, \qquad Q' = 0, \\ f(0) = f''(0) = G'(0) = 0, \qquad f(1) = f'(1) = G(1) = 0. \end{gathered} \tag{3.3}$$

We note that, for $\lambda = \lambda_0$ and $\alpha = 0$, system (3.3) has the solution $f = 0$, $G = \varphi(\xi)$, and $Q = 0$, since it coincides with the eigenvalue problem (3.1).

We now consider the Cauchy problem

$$L_1(f, G, Q) = 0, \qquad K_1(f, G) = 0, \qquad Q' = 0; \tag{3.4}$$

$$f = 0, \quad f' = p1, \quad f'' = 0, \quad G = 1, \quad G' = 0, \quad Q = p2 \quad (\xi = 0). \tag{3.5}$$

The parameters $p1$ and $p2$ are not yet known and are found when the boundary conditions in (3.3) on the rigid wall are satisfied for $\xi = 1$.

We note that, for $\lambda = \lambda_0$ and $\alpha = 0$, problem (3.4), (3.5) has the solution $f = 0$, $G = \varphi(\xi)$, $Q = 0$, and $p1 = p2 = 0$. We now study the solution of this problem for values of (λ, α) close to $(\lambda_0, 0)$. Obviously, this solution is the solution of the boundary-value problem (3.3) if and only if the functions f, G, and Q satisfy the boundary conditions for $\xi = 1$:

$$f(1,\lambda,\alpha,p1,p2) = 0, \qquad f'(1,\lambda,\alpha,p1,p2) = 0, \qquad G(1,\lambda,\alpha,p1,p2) = 0. \tag{3.6}$$

The parameters $p1$ and $p2$ are uniquely determined from the first two equations of system (3.6). This is established by means of the known theorem of implicit functions [5]. At the point $\lambda = \lambda_0$ and $\alpha = 0$, the calculation gave a nonzero functional determinant $D(f, f')/D(p1, p2)$. In this calculation of the determinant, the Cauchy problems for the derivatives of the functions f and $\partial f/\partial\xi$ with respect to the parameters $p1$ and $p2$ were solved numerically.

Having determined the parameters $p1$ and $p2$ from the first two equations of system (3.6) and substituting them into the third equations, we derive the bifurcation equation

$$b(\lambda,\alpha) \equiv G(1,\lambda,\alpha,p1(\lambda,\alpha),p2(\lambda,\alpha)) = 0. \tag{3.7}$$

Using the method of [4], we expand the function $b(\lambda,\alpha)$ into the Taylor finite series in the neighborhood of the point $\lambda = \lambda_0$ and $\alpha = 0$:

$$b(\lambda,\alpha) = b(\lambda_0,0) + (\lambda-\lambda_0)b_\lambda + \alpha b_\alpha + 0.5\alpha^2 b_{\alpha\alpha} + \ldots = 0. \tag{3.8}$$

Here b_λ, b_α, and $b_{\alpha\alpha}$ are the derivatives of the function $b(\lambda,\alpha)$, with respect to the parameters λ and α which were calculated at the point $\lambda = \lambda_0$, $\alpha = 0$.

We calculate the coefficients of the series (3.8). We show that $b(\lambda_0, 0) = 0$. To do this, we pass to the limit for $\lambda \to \lambda_0$ and $\alpha \to 0$ in (3.7) and to the Cauchy problem (3.4), (3.5) and take into account the conditions $f = 0$, $f' = 0$, and $G = 0$ for $\xi = 1$. Analysis of the problems obtained shows that $G = \varphi(\xi)$ and $p1 = p2 = 0$ for $\alpha = 0$ and $\lambda = \lambda_0$, and, hence, $b(\lambda_0, 0) = \varphi(1) = 0$.

The numerical value of the coefficient b_λ in (3.8) is found using the relation

$$b_\lambda = \frac{\partial G}{\partial\lambda} + \frac{\partial G}{\partial p1}\frac{\partial p1}{\partial\lambda} + \frac{\partial G}{\partial p2}\frac{\partial p2}{\partial\lambda} \quad (\lambda = \lambda_0,\ \alpha = 0).$$

It is noteworthy that $\partial G/\partial p1 = 0$ and $\partial G/\partial p2 = 0$ ($\lambda = \lambda_0$ and $\alpha = 0$). This is found by studying the Cauchy problems obtained by differentiating problem (3.4), (3.5) with respect to the parameters $p1$ and $p2$. Differentiating (3.4) and (3.5) with respect to λ and letting $\lambda \to \lambda_0$ and $\alpha \to 0$, we derive the following Cauchy problem for $G_\lambda \equiv \partial G/\partial\lambda$:

$$K_0 G_\lambda = 2\lambda_0(\Phi'_{0\lambda}G - \Phi_{0\lambda}G') + 2(\Phi'_0\varphi - \Phi_0\varphi'), \qquad G_\lambda = 0, \qquad G'_\lambda = 0 \quad (\xi = 0).$$

The function $\Phi_{0\lambda}$ is determined from the boundary-value problem

$$L_0\Phi_{0\lambda} - \lambda_0 q_{0\lambda} = \Phi_0'^2 - 2\Phi_0\Phi_0'' + q_0, \quad q'_{0\lambda} = 0,$$

$$\Phi_{0\lambda} = 0, \quad \Phi''_{0\lambda} = 0 \quad (\xi = 0), \qquad \Phi_{0\lambda} = \Phi'_{0\lambda} = 0 \quad (\xi = 1).$$

A numerical calculation yields $q_{0\lambda} = 0.0108$, $\Phi'_{0\lambda}(0) = -0.0024$, and $G_\lambda(1) = -0.1629$. Thus, we have $b_\lambda = G_\lambda(1, \lambda_0, 0) = -0.1629$.

We note that $b_\alpha = 0$ in relation (3.8). This is established when the Cauchy problems are studied for the derivatives $\partial G/\partial\alpha$, $\partial G/\partial p1$, and $\partial G/\partial p2$.

The coefficient $b_{\alpha\alpha}$ in (3.8) is found taking into account that $b_{\alpha\alpha} = G_{\alpha\alpha}$ for $\lambda = \lambda_0$ and $\alpha = 0$. The function $G_{\alpha\alpha}$ is found by numerically solving the Cauchy problem

$$K_0 G_{\alpha\alpha} = 4\lambda_0(f'_\alpha\varphi - f_\alpha\varphi') \quad (\lambda = \lambda_0,\ \alpha = 0), \qquad G_{\alpha\alpha} = 0, \quad G'_{\alpha\alpha} = 0 \quad (\xi = 0).$$

The boundary-value problem for f_α is obtained by replacing the functions f_1 and q_1 by f_α and $Q_\alpha + \varphi^2$, respectively, in (3.1). A numerical calculation gives $G_{\alpha\alpha} = 6.6987$ ($\xi = 1$, $\lambda = \lambda_0$, and $\alpha = 0$).

Using Newton's diagram [5], one can find the parameter α from the bifurcation equation (3.8):

$$\alpha = \pm(2b_\lambda b_{\alpha\alpha}^{-1}(\lambda_0 - \lambda))^{1/2} + \ldots \quad (\lambda \to \lambda_0).$$

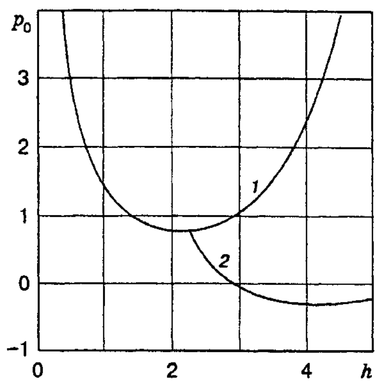

Fig. 1

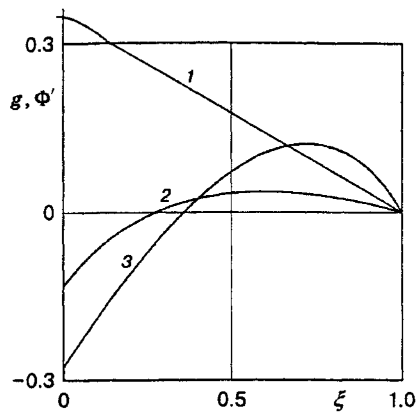

Fig. 2

As a result, two symmetrical solutions, which differ from each other only by the direction of the circumferential velocity component and exist for all $h > h_0 = 2.2388$ ($\lambda > \lambda_0$), bifurcate from the solution Φ_0 and q_0 at the point $\lambda = \lambda_0$.

The asymptotic behavior of the bifurcating solutions is constructed as $\lambda \to \lambda_0$. Introducing the small parameter $\varepsilon_1 = (\lambda - \lambda_0)^{1/2}$, we present the solution of problem (2.1) as the series

$$\Phi = \Phi_{00}(\xi) + \varepsilon_1 F_1 + \varepsilon_1^2(\Phi_{02} + F_2) + \ldots,$$
$$g = \varepsilon_1 G_1 + \varepsilon_1^2 G_2 + \varepsilon_1^3 G_3 + \ldots, \qquad q = q_{00} + \varepsilon_1 Q_1 + \varepsilon_1^2(q_{02} + Q_2) + \ldots . \tag{3.9}$$

The representation of the functions $\Phi_0(\xi, \lambda)$ and $q_0(\lambda)$ is used here in the form of series with respect to the degrees of the parameter ε_1; the coefficients of these series are found from the relations $\Phi_{00} = \Phi_0(\xi, \lambda_0)$, $q_{00} = q_0(\lambda_0)$, $\Phi_{02} = \partial\Phi_0/\partial\lambda$, and $q_{02} = \partial q_0/\partial\lambda$ ($\lambda = \lambda_0$).

The series (3.9) are substituted into system (2.1) and the coefficients at $\varepsilon_1, \varepsilon_1^2, \ldots$ are equated to zero. For F_1, G_1, and Q_1, an eigenvalue problem is derived. This problem is obtained from (3.1) by replacing the functions f_1, g_1, and q_1 by F_1, G_1, and Q_1, respectively. The latter problem is solved in the form $F_1 = Q_1 = 0$ and $G_1 = c_1\varphi(\xi)$, where φ is the normalized eigenfunction determined above, and c_1 is the constant determined from the solvability conditions for the boundary-value problem in the third approximation. Now we derive boundary-value problems for the functions F_2, G_2, and Q_2:

$$L_0 F_2 = \lambda_0(Q_2^2 - G_1^2), \qquad K_0 G_2 = 0, \qquad Q_2' = 0,$$
$$F_2 = F_2'' = G_2' = 0 \quad (\xi = 0), \qquad F_2 = F_2' = G_2 = 0 \quad (\xi = 1).$$

A numerical calculation yields $Q_2 = 0.7183c_1^2$ and $F_2'(0) = 0.3629c_1^2$. For G_2, we obtain $G_2 = c_2\varphi$, where the constant c_2 is determined by studying the solvability conditions for the boundary-value problems in the fourth approximation. Writing the boundary-value problem for the function G_3 and satisfying its solvability condition, we find the constant c_1:

$$c_1^2 = -I_1/I_2, \qquad I_2 = \lambda_0 \int_0^1 \varphi_*(\tilde{F}_2'\varphi - \tilde{F}_2\varphi')\,d\xi,$$

$$I_1 = \int_0^1 \varphi_*(\Phi_{00}'\varphi - \Phi_{00}\varphi' + \lambda_0\Phi_{02}'\varphi - \lambda_0\Phi_{02}\varphi')\,d\xi.$$

Here $\tilde{F}_2 = F_2 c_1^{-2}$, and φ_* is determined from the boundary-value problem $K_*\varphi_* = 0$, $\varphi_*'(0) = \varphi_*(1) = 0$, where $K_* = D^2 - 2\lambda_0(2\Phi_{00}'I + \Phi_{00}D)$ is the operator conjugate to K_0. A numerical calculation yields $c_1^2 = 0.0486$.

For finite values of the difference $\lambda - \lambda_0$ (for all $\lambda > \lambda_0$), the bifurcated solutions are constructed by numerically integrating system (2.1). We note that these solutions show only different signs of the function $g(\xi, \lambda)$. This means that one bifurcated solution is obtained from another by changing the direction of the circumferential velocity component v_θ, whereas $v_\theta = 0$ for the "basic" solution. Curve 2 in Fig. 1 shows the dependence of $p_0 = -qh^2$ on the layer thickness h for a bifurcating solution. Curves 1 and 2 in Fig. 2 show, respectively, the functions $g(\xi)$ and $\Phi'(\xi)$ for bifurcating solutions, and curve 3 refers to the diagram $\Phi'(\xi)$ for the "basic" solution ($g = 0$) when $\lambda = 27$, which corresponds to $h = 3$. Evidently, the radial velocity component changes the direction inside the layer once, whereas, monotonically decreasing from the free boundary to the rigid wall, the circumferential component preserves its direction.

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